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A geometrical approach to the problem of integrability of Hamiltonian systems by separation of variables

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Abstract

We propose a geometrical approach to the problem of integrability of Hamiltonian systems of low dimensions using the Hamilton–Jacobi method of separation of variables, based on the method of moving frames. As an illustration we present a complete classification of all separable Hamiltonian systems defined in two-dimensional Riemannian manifolds of arbitrary curvature and a criterion for separability. Connections to bi-Hamiltonian theory are also found. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

We are concerned with a general Hamiltonian system defined by the Hamiltonian function H_0:

$$H_0 = \frac{1}{2} g^{ij}(\mathbf{q}) p_i p_j + V(\mathbf{q}), \quad i, j = 1, \dots, n. \tag{1.1}$$

This implies that the Hamiltonian vector field X_{H_0} corresponding to (1.1) is defined with respect to the canonical symplectic structure $\boldsymbol{\omega}_0 = \sum_{i=1}^n \mathrm{d}p^i \wedge \mathrm{d}q_i$, or Poisson bi-vector

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$\mathbf{P}_0 = \sum_{i=1}^{n} \partial_i \wedge \partial^i$ in the usual way:

$$X_{H_0} = [\mathbf{P}_0, H_0]. \tag{1.2}$$

Here and below, unless otherwise indicated, [,] denotes the Schouten bracket [1] which generalizes the usual Lie bracket of vector fields and $\partial_i = \partial/\partial q^i$, $\partial^i = \partial/\partial p_i$. The quantities $g^{ij}, i, j = 1, \dots, n$ represent a (2, 0) metric tensor $\mathbf{g}$. Thus, in addition to being defined in a $2n$-dimensional symplectic (Poisson) manifold $(M, \boldsymbol{\omega}_0)((M, \mathbf{P}_0))$ the system (1.1) is also defined in an n-dimensional pseudo-Riemannian manifold $(\tilde{M}, \mathbf{g})$, where M is obviously the cotangent bundle of $\tilde{M}$: $M = T\tilde{M}^*$. Hence, the (q^i, p_i) are the canonical coordinates in $T\tilde{M}^*$. Conceivably, this arrangement provides a rich geometrical background for the study of the Hamiltonian system (1.1). A fundamental question is whether the system (1.1) is completely or Liouville integrable, or, in other words, whether the system (1.1) possesses n functionally independent first integrals in involution with respect to the Poisson bracket defined by $\boldsymbol{\omega}_0$ (or $\mathbf{P}_0$). If the answer is positive, then according to the celebrated Arnol'd–Liouville theorem the Hamiltonian system (1.1) can be integrated by quadratures [2].

In spite of the recent development of methods and techniques of complete integrablity that have been invented in the last three decades (i.e., the method of Lax pairs, the bi-Hamiltonian method, etc.), the classical 19th century approach to complete integrability via the Hamilton–Jacobi method of separation of variables is being revived. One of the main impetuses for the renewed interest in this method was Carter's discovery that the geodesic equations in the Kerr black hole space–time can be integrated by separation of variables [3]. Remarkably, in the course of the last 10 years, this classical method has been effectively linked with the method of the Lax representation and the bi-Hamiltonian method, thus leading to new theories in the area of integrable Hamiltonian systems (see [4,5], respectively, and the relevant references therein).

The key idea behind the method of separation of variables is to seek a set of special coordinates $\mathbf{q} := (q^1, \dots, q^n)$ in which the corresponding Hamilton–Jacobi partial differential equation

$$\tfrac{1}{2} g^{ij} \partial_i W \partial_j W + V = E \tag{1.3}$$

admits a complete integral of the form

$$W(\mathbf{q}, \mathbf{c}) = W_1(q^1, \mathbf{c}) + \cdots + W_n(q^n, \mathbf{c}), \tag{1.4}$$

where $\mathbf{c} = (c_1, \dots, c_n)$ are the constants of integration. These constants are the n first integrals in involution with respect to $\boldsymbol{\omega}_0$ (or $\mathbf{P}_0$) that guarantee the complete integrability of (1.1). A complete integral W can be interpreted as an n-dimensional Lagrangian submanifold in M lying on the level surface $H_0 = \text{const}$. The coordinates $(q^1, \dots, q^n)$ in (1.4) are called *separable coordinates*. Moreover, if the metric $\mathbf{g}$ of (1.1) is diagonal in these coordinates, they are also said to be *orthogonal* and the system defined by the Hamiltonian (1.1) is said to be *orthogonally separable*. In what follows, we concentrate our attention on this type of separable Hamiltonian systems. We note that the orthogonal case has been extensively studied in the past in numerous articles by such famous scholars as Dall'Acqua, Eisenhart, Levi-Civita, Ricci, Stäckel and others. Major advances in the area have been achieved in recent years by Benenti, Kalnins and Miller, Shapovalov, as well as many others. For a complete list of references see [6,7]. The non-orthogonal case has also received much attention [6].

The main objective of this paper is to combine the theory of the orthogonally separable Hamiltonian systems and the method of moving frames. The method has been extensively studied and successfully applied under different names (for instance, "the method of quasi-coordinates", "... non-coordinate basis", "... orthogonal ennuples") in such areas of mathematics and physics as differential geometry, general relativity and theory of Lie groups. Introduced by Darboux and developed by Cartan, the method has been chiefly used in two cases: As an alternative method to the classical tensor calculus to avoid, in Cartan's words [8], the "debauch d'indices" and as an effective tool to study geometrical invariants of submanifolds under the action of transformation Lie groups. We refer the interested reader to a recent review by Olver [10] where the latest advances and a complete list of references pertinent to the latter case can be found. In the present work, we are mainly concerned with the former case, when the application of the moving frames method can significantly alleviate the complications of dealing with tensorial geometrical quantities, in this paper, defined in a Riemannian manifold $(\tilde{M}, \mathbf{g})$. We note that an equivalent version of the method of moving frames based on the frame of vectors, unlike Cartan's approach via co-vectors, was effectively used by Eisenhart [9].

The essence of the method of moving frames can be briefly described as follows. In a given n-dimensional pseudo-Riemannian manifold $(\tilde{M}, \mathbf{g})$ at each point $p \in \tilde{M}$ we replace for the natural basis of the cotangent space $T\tilde{M}_p^*$: $(\mathrm{d}q^1, \dots, \mathrm{d}q^n)$ arising from a coordinate system $(q^1, \dots, q^n)$ by a basis of n pointwise linearly independent one-forms (co-vectors) $E^1, \dots, E^n \in T\tilde{M}_p^*$, that can be *adapted to the geometric situation*. In the considerations that follow the natural choice is that in which the metric tensor $\mathbf{g}$ takes its algebraic canonical form. In other words, with respect to the basis E^a, $a = 1, \dots, n$, we have

$$g_{ab} = \mathrm{diag}(1, \dots, 1, -1, \dots, -1). \tag{1.5}$$

The co-frame of one-forms $E^1, \dots, E^n$ is said to be rigid in this case. One can now proceed to study the relations between the one-forms $E^a \in T\tilde{M}_p^*$, their exterior derivatives $\mathrm{d}E^a$ and the dual basis $(E_1, \dots, E_n)$ of the tangent space $T\tilde{M}_p$ *independently of local coordinates*. Thus, we can consider an open set $A \ni p$ and (*orthonormal*) *moving co-frame* $E^1, E^2, \dots, E^n$ of one-forms defined in A for which the metric tensor $\mathbf{g}$ takes the form (1.5). We note that the elements of the moving co-frame E^a and their counterparts E_a are connected with the natural basis associated to local coordinates $(q^1, \dots, q^n)$ about $p \in A$ as follows:

$$E^a = h_i^a \,\mathrm{d}q^i, \qquad E_a = h_a{}^i \frac{\partial}{\partial q^i}. \tag{1.6}$$

The structure functions C_{ab}^c are defined by

$$[E_a, E_b] = C_{ab}^c E_c \quad \text{or} \quad \mathrm{d}E^a = -\tfrac{1}{2} C_{bc}^a E^b \wedge E^c. \tag{1.7}$$

Now by (1.6) $C^c_{ab} = h^c_i(h_a{}^j h_{b,j}{}^i - h_b{}^j h_{a,j}{}^i)$, $a, b, c, i, j = 1, \dots, n$. Here and below, i denotes the usual partial derivative with respect to the ith coordinate. We introduce the connection coefficients Γ corresponding to the Levi-Civita connection ∇ associated to g_{ab} as follows:

$$\nabla_{E_a} E_b = \Gamma_{ab}{}^c E_c, \qquad \nabla_{E_c} E^b = -\Gamma_{cd}{}^b E^d.$$

The vanishing of the torsion tensor of ∇ is expressed by

$$\Gamma_{bc}{}^a - \Gamma_{cb}{}^a - C^a_{bc} = 0, \tag{1.8}$$

while the curvature tensor of ∇ is given by

$$R^a_{bcd} = E_c \Gamma_{db}{}^a + \Gamma_{db}{}^e \Gamma_{ce}{}^a - E_d \Gamma_{cb}{}^a - \Gamma_{cb}{}^e \Gamma_{de}{}^a - C^e_{cd} \Gamma_{eb}{}^a. \tag{1.9}$$

We now define a one-form valued matrix ω^a_b called the *connection one-form* by

$$\omega^a_b := \Gamma_{cb}{}^a E^c. \tag{1.10}$$

Further, we define

$$\omega_{ab} := g_{ac} \omega^c_b.$$

On account of the above connection one-forms, ω_{ab} are obviously skew-symmetric. The condition (1.8) and the definition (1.9) may be expressed in the language of differential forms as

$$\mathrm{d}E^a + \omega^a_b \wedge E^b = 0, \tag{1.11}$$

and

$$\mathrm{d}\omega^a_b + \omega^a_c \wedge \omega^c_b = \Theta^a_b, \tag{1.12}$$

where $\wedge$ is exterior multiplication, d the exterior derivative and $\Theta^a_b := \frac{1}{2} R^a_{bcd} E^c \wedge E^d$ the *curvature two-form*. Taking the exterior derivative of (1.11) and (1.12) yields the first and second Bianchi identities, respectively

$$\Theta^a_b \wedge E^b = 0, \tag{1.13}$$

and

$$\mathrm{d}\Theta^a_b + \omega^a_c \wedge \Theta^c_b - \Theta^a_c \wedge \omega^c_b = 0. \tag{1.14}$$

Finally, the equations satisfied by a valence two, symmetric, covariant Killing tensor $\mathbf{K}$ can be written in frame components as

$$K_{(ab;c)} = 0, \tag{1.15}$$

where ; denotes the covariant derivative defined by

$$K_{ab;c} := E_c K_{ab} - K_{db} \Gamma_{ca}{}^d - K_{ad} \Gamma_{cb}{}^d. \tag{1.16}$$

This is all the geometric machinery that we need in the forthcoming sections to study integrability of Hamiltonian systems by the method of separation of variables.

2. Orthogonal separability

As we mentioned in Section 1, the question of whether the Hamiltonian system defined by (1.1) is orthogonally separable has a long history.

It was Stäckel who first found in 1893 the necessary and sufficient conditions for the system (1.1) to be orthogonally separable [11]. In spite of their rather complicated form these fundamental conditions are still being used today by many mathematicians to study orthogonal separability.

Levi-Civita [12] established a (local) criterion of separability (not necessarily orthogonal) of the Hamilton–Jacobi equation associated with a general Hamiltonian system defined by (1.1) in local coordinates $(q^1, \dots, q^n; p_1, \dots, p_n)$ consisting of the $1/2n(n-1)$ equations

$$\partial^i \partial^j H \partial_i H \partial_j H - \partial_i \partial^j H \partial^i H \partial_j H + \partial_i \partial_j H \partial^i H \partial^j H - \partial^i \partial_j H \partial_i H \partial^j H = 0. \quad (2.1)$$

The next breakthrough was by Eisenhart [13] who presented in turn necessary and sufficient conditions for a Hamiltonian system defined by the geodesic Hamiltonian

$$H_g = \tfrac{1}{2} g^{ij} p_i p_j \quad (2.2)$$

to be of the Stäckel type and thus orthogonally integrable. The result was based on the fact that the n first integrals in involution (including the Hamiltonian) are necessarily quadratic in momenta, when the system defined by (2.2) is considered in the natural position–momenta coordinates. Moreover, the involution of any of these $n-1$ first integrals $F_1, \dots, F_{n-1}$:

$$F_r := \tfrac{1}{2} K_r^{ij} p_i p_j, \quad r = 1, \dots, n-1$$

with the Hamiltonian (2.2):

$$\{H_g, F_r\} = 0, \quad r = 1, \dots, n-1$$

entails the Killing tensor equation

$$[\mathbf{g}, \mathbf{K}_r] = 0, \quad r = 1, \dots, n-1,$$

which is equivalent to

$$K_{r(ab;c)} = 0, \quad r = 1, \dots, n-1,$$

where the indices of $\mathbf{K}_1, \dots, \mathbf{K}_{n-1}$ have been lowered. Hence, the first integrals $F_1, \dots, F_{n-1}$ are defined by the $n-1$ valence two Killing tensors $\mathbf{K}_1, \dots, \mathbf{K}_{n-1}$ that share, in view of Eisenhart's result, certain geometrical properties. In particular, they must possess the same eigenvectors and these eigenvectors are *normal*, which means that each eigenvector is normal to an $(n-1)$-dimensional hypersurface.

Kalnins and Miller [14] have further improved the results of Eisenhart. In particular, they have studied the n-dimensional Abelian Lie algebra of Killing tensors of order 2, $\tilde{\mathbf{K}}_1, \dots, \tilde{\mathbf{K}}_n$, where $\tilde{\mathbf{K}}_1 = \mathbf{g}, \dots, \tilde{\mathbf{K}}_n = \mathbf{K}_{n-1}$ in the notation above. Indeed, we note that the Schouten bracket satisfies the Jacobi identity in the space of two-contravarant tensors (symmetric or otherwise). Moreover, they concluded that *every Killing tensor* $\tilde{\mathbf{K}}_i$, $i = 2, \dots, n$

that is linearly independent of $\mathbf{g} = \tilde{\mathbf{K}}_1$ *and satisfies the properties of the main Theorem 6* (*see* [14]) *defines* (*locally*) *a separable coordinate system for the Hamilton–Jacobi equation* (1.3) *on* $(\tilde{M}, \mathbf{g})$, *and conversely, every separable coordinate system arises in this way*. We note, however, that the complications arising from dealing with the n Killing tensors (including the metric) connected via certain algebraic and differential conditions makes this result difficult to apply.

Finally, Benenti [6,15] generalized the results above and obtained a characterization of orthogonal separability in terms of a *single* Killing tensor. His result is the following theorem.

Theorem 2.1 (Benenti). *A Hamiltonian system defined by* (1.1) *is orthogonally separable if and only if there exists a valence two Killing tensor* $\mathbf{K}$ *with pointwise simple and real eigenvalues, orthogonally integrable eigenvectors and such that* $\mathrm{d}(\hat{\mathbf{K}}\,\mathrm{d}V) = 0$, *where the linear operator* $\hat{\mathbf{K}}$ *is given by* $\hat{\mathbf{K}} := \mathbf{K}\mathbf{g}$ (*or in the index form* $\hat{K}^i_j := K^{i\ell} g_{\ell j}$).

Remark 2.2. We note that starting with one $\mathbf{K}$ that satisfies the conditions of Theorem 2.1, one can reconstruct the n-dimensional Abelian Lie algebra of Killing tensors (including the metric) of Theorem 6 in [14] by either finding the sets of separable coordinates or using the intrinsic iterative process described in [16], which does not require having separable coordinates. Conversely, having the n-dimensional Abelian Lie algebra, we can easily obtain the Killing tensor $\hat{\mathbf{K}}$ of Theorem 2.1 by considering the total sum of its elements. Another way to see this is the following: The Killing equation (3.2) for $\hat{\mathbf{K}}$ is equivalent to a system of n linear partial differential equations, the general solution of which naturally depends on n constants of integration, where in turn can be viewed as the dimension of the corresponding Abelian Lie algebra of Killing tensors. Further, the Killing tensor $\mathbf{K}$ does not define a single set of separable coordinates, for example, by varying its eigenvalues (i.e., intrinsic invariants) or otherwise [17], we can extract *all* the sets of orthogonally separable coordinates for a given Hamiltonian system defined by (1.1).

Remark 2.3. The statement of Theorem 2.1 implies that there exists an additional first integral quadratic in momenta (say):

$$F(\mathbf{q}, \mathbf{p}) = \tfrac{1}{2} K^{ij}(\mathbf{q}) p_i p_j + U(\mathbf{q}), \tag{2.3}$$

where the matrix K^{ij} is that of $\mathbf{K}$. The involutiveness $\{H_0, F\} = 0$ yields the Killing equation $[\mathbf{g}, \mathbf{K}] = 0$, and the condition $\mathrm{d}(\hat{\mathbf{K}}\,\mathrm{d}V) = 0$ (which entails locally that $\mathrm{d}U = \hat{\mathbf{K}}\,\mathrm{d}V$).

Theorem 2.1 offers the advantage of working with a single geometrical quantity instead of n such quantities as in [14]. However, in general it is still very difficult to check whether or not a given Killing tensor $\mathbf{K}$ has normal eigenvectors. This is a rather non-trivial task even in three-dimensional pseudo-Riemannian manifolds $(\tilde{M}, \mathbf{g})$. The main difficulty is the computational effort required by the straightforward approach. To solve the Killing equation in this case in given position–momenta coordinates yields six functions (i.e.,

K^{11}, K^{22}, K^{33}, K^{12}, K^{23}, K^{13}) depending upon 20 constants of integration that represent the dimension of the space of (2, 0) Killing tensors in $\mathbb{R}^3$ (see [18–20]). Conceivably, for $n = 4$, where $n = \dim \tilde{M}$ the problem of finding the normal eigenvectors of $\mathbf{K}$ is practically insurmountable without employing computer algebra.

Therefore, in this paper, we propose the use of the moving frame approach where the frame vectors are chosen to be a set of suitably normalized eigenvectors of $\mathbf{K}$. It appears that the method not only results in a significant algebraic simplification, but also allows one to consider the problem in a much more general setting, namely *without any restrictions at all on the curvature of the pseudo-Riemannian manifold* $(\tilde{M}, \mathbf{g})$.

To demonstrate how the method works and give a flavor of its applications, we begin by proving the following criterion for orthogonal separability in Cartesian coordinates.

Theorem 2.4. *The Hamiltonian system* (1.1) *is orthogonally separable with respect to Cartesian coordinates iff the associated pseudo-Riemannian manifold* $(\tilde{M}, \mathbf{g})$ *admits a valence two covariant Killing tensor* $\mathbf{K}$ *with pointwise simple eigenvalues and vanishing Nijenhuis tensor* $\mathbf{N}_{\hat{\mathbf{K}}}$.

Proof. Consider a C^∞ pseudo-Riemannian manifold $(\tilde{M}, \mathbf{g})$ associated to the Hamiltonian (1.1) which possesses a symmetric C^∞ tensor field $\mathbf{K}$ of type (0, 2). The eigenvalue equation

$$K_{ij}E^j = \lambda g_{ij}E^j \tag{2.4}$$

admits n pointwise simple eigenvalues $\lambda_1, \ldots, \lambda_n$. We note that since $(\tilde{M}, \mathbf{g})$ is the Riemannian eigenvalues are necessarily real. Let $E_1, \ldots, E_n$ be a set of eigenvectors of $\mathbf{K}$ corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_n$. It can be shown that the eigenvectors are real, mutually orthogonal and that none of them is a null vector. Thus, the eigenvectors can be normalized such that

$$\mathbf{g}(E_a, E_a) = 1. \tag{2.5}$$

The above set of eigenvalues is uniquely determined up to sign.

Since $\mathbf{g}$ and $\mathbf{K}$ are C^∞ tensor fields, and the operations of solving for the eigenvalues and eigenvectors and normalizing the eigenvectors are rational operations it follows that the eigenvectors $E_1, \ldots, E_n$ define a set of C^∞ pointwise linearly independent vector fields on some open set $A \subset \tilde{M}$. Hence, we may choose these vectors as a rigid moving frame on A with respect to which the components of $\mathbf{g}$ and $\mathbf{K}$ are given by

$$g_{ab} = \mathrm{diag}(1, \ldots, 1), \tag{2.6}$$

and

$$K_{ab} = \mathrm{diag}(\lambda_1, \ldots, \lambda_n). \tag{2.7}$$

It follows that the metric tensor has the form

$$\mathrm{d}s^2 = (\mathrm{d}x^1)^2 + \cdots + (\mathrm{d}x^n)^2. \tag{2.8}$$

A rigid co-frame can thus be chosen as follows:

$$E^1 = \mathrm{d}x^1, \ldots, E^n = \mathrm{d}x^n$$

with corresponding dual frame being

$$E_1 = \partial_i, \ldots, \partial_n. \tag{2.9}$$

It is obvious that the frame vector fields are orthogonally integrable. Consider now the (0, 2) tensor **K**, the components of which in the above co-frame are given by

$$K_{ab} = \mathrm{diag}(\lambda_1, \ldots, \lambda_n) \tag{2.10}$$

with λ_a are constants satisfying $\lambda_a \neq \lambda_b$ for all $a, b = 1, \ldots, n, a \neq b$. It is clear that E_a is an eigenvector corresponding to the eigenvalue λ_a for each $a = 1, \ldots, n$. Since the connection coefficients for the frame (2.9) are zero, Eq. (1.16) has the form

$$K_{ab;c} = \partial_c K_{ab}.$$

It is thus easy to verify that the tensor (2.10) satisfies the Killing equation (1.15). We conclude that the tensor defined by (2.10) is the Killing tensor, the existence of which is guaranteed by Theorem 1.1. It follows from (2.8) and (2.10) that

$$\hat{\mathbf{K}} = \mathrm{diag}(\lambda_1, \ldots, \lambda_n), \tag{2.11}$$

and that $\hat{K}$ has a trivially vanishing Nijenhuis tensor [21]. This fact may be established from the following expression of $\mathbf{N}_{\hat{\mathbf{K}}}$ in local coordinates:

$$N^i_{\hat{\mathbf{K}}jk} = \partial_\ell B^i_k B^\ell_j - \partial_\ell B^i_j B^\ell_k + \partial_k B^\ell_j B^i_\ell - \partial_j B^\ell_k B^i_\ell = 0, \tag{2.12}$$

where $i, j, k = 1, \ldots, n$. Note that $N^i_{\hat{\mathbf{K}}jk} = -N^i_{\hat{\mathbf{B}}kj}$.

Let **K** be a (0, 2) Killing tensor with pointwise simple and real eigenvalues and vanishing Nijenhuis tensor. In the rigid moving frame of eigenvectors $E_1, \ldots, E_n$ of **K** the condition (2.12) reads

$$N_{\hat{\mathbf{K}}}(E_a, E_b) = (\hat{\mathbf{K}} - \lambda_a)(\hat{\mathbf{K}} - \lambda_b)C^c_{ab}E_c + (\lambda_a - \lambda_b)(E_a(\lambda_b)E_b + E_b(\lambda_a)E_a) = 0, \tag{2.13}$$

and taking into account (2.11) can be decomposed into the following system of equations:

$$C^c_{ab} = 0, \quad a, b, c \text{ are distinct}, \tag{2.14}$$

$$E_a(\lambda_b)(\lambda_a - \lambda_b) = 0, \quad a, b \text{ are distinct}. \tag{2.15}$$

Concurrently, the Killing equation (1.15) for **K** with lower indices decomposes as follows:

$$K_{(aa;a)} = 0 \Leftrightarrow E_a K_{aa} = 0 \quad n \text{ equations}, \tag{2.16}$$

$$K_{(aa;b)} = 0 \Leftrightarrow E_b(\lambda_a) = 2\Gamma_{aab}(\lambda_b - \lambda_a) \quad 2\binom{n}{2} \text{ equations}, \tag{2.17}$$

and

$$K_{(ab;c)} = 0 \quad \binom{n}{3} \text{ equations,} \tag{2.18}$$

where a, b and c are distinct. Therefore, in view of the above since $\lambda_1, \ldots, \lambda_n$ are distinct, the connection coefficients Γ^a_{bc} vanish. Hence, the Riemannian space $(\tilde{M}, \mathbf{g})$ is flat and the eigenvalues of $\mathbf{K}$ are constants. This implies that the Hamiltonian system defined by (1.1) is separable only with respect to Cartesian coordinates. □

Remark 2.5. It is instructive to contrast the above result with an analogous result for Poisson–Nijenhuis manifolds. Recall that in the case of two compatible Poisson bi-vectors $\mathbf{P}_1$ and $\mathbf{P}_2$, the linear operator $\mathbf{A} := \mathbf{P}_2\mathbf{P}_1^{-1}$ with the components $A^i_j = P_2^{im}P^{-1}_{1mj}$ (if $\mathbf{P}_1$ is non-degenerate) has a vanishing Nijenhuis tensor $\mathbf{N}_\mathbf{A} = 0$ and vice versa (see, for example, [22–24]). We observe that the Killing tensor equation $[\mathbf{g}, \mathbf{K}] = 0$ satisfied by the two Killing tensors $\mathbf{g}$ and $\mathbf{K}$ resembles the condition $[\mathbf{P}_1, \mathbf{P}_2] = 0$ of Compatibility of the two Poisson bi-vectors in the theory of bi-Hamiltonian systems. However, as may be seen from the proof of Theorem 2.4, the Killing tensor equation is *not* equivalent to the vanishing of the Nijenhuis tensor of the corresponding linear operator $\hat{\mathbf{K}} := \mathbf{K}\mathbf{g}$. Moreover, as we have just seen, the vanishing of the tensor $\mathbf{N}_{\hat{\mathbf{K}}}$ appears to be a very restrictive additional condition on $\hat{\mathbf{K}}$.

3. Separability in two-dimensional Riemannian manifolds

We start our considerations in an *arbitrary* Riemannian manifold $(\tilde{M}, \mathbf{g})$, $\dim \tilde{M} = 2$ defined by (1.1) making a priori no assumptions on its curvature. Using the techniques presented in the previous two sections, we introduce a rigid moving frame of co-vectors E^1, E^2 with respect to which the metric $\mathbf{g}$ and Killing tensor $\mathbf{K}$ of Theorem 2.1 take the following forms:

$$g_{ab} = \delta_{ab}E^a \odot E^b, \tag{3.1}$$

$$K_{ab} = \lambda_a\delta_{ab}E^a \odot E^b, \tag{3.2}$$

where $\odot$ is the symmetric tensor product and $a, b = 1, 2$ and λ_1, λ_2 along with the dual vectors E_1, E_2 are the eigenvalues and eigenvectors of $\mathbf{K}$, respectively. In this case we have two independent connection coefficients Γ_{112} and Γ_{212} and one component of the Riemann curvature tensor R_{1212}. For convenience we write $\alpha := \Gamma_{112}$ and $\beta := \Gamma_{212}$. Then the formulas (1.7), (1.9) and (1.16) become

$$[E_1, E_2] = -\alpha E_1 - \beta E_2, \tag{3.3}$$

$$\mathrm{d}E^1 = \alpha E^1 \wedge E^2, \qquad \mathrm{d}E^2 = \beta E^1 \wedge E^2, \tag{3.4}$$

$$R_{1212} = -E_1\beta + E_2\alpha - \alpha^2 - \beta^2, \tag{3.5}$$

$$E_1\lambda_1 = 0, \qquad E_2\lambda_1 = 2\alpha(\lambda_2 - \lambda_1), \qquad E_1\lambda_2 = 2\beta(\lambda_2 - \lambda_1), \qquad E_2\lambda_2 = 0, \tag{3.6}$$

where (1.8) has been used. Our next observation is that in a two-dimensional Riemannian manifold the conditions of orthogonal integrability for E_1 and E_2, $E^a \wedge \mathrm{d}E^a = 0, a = 1, 2$ are automatically satisfied. Hence, by Frobenius' theorem, there exist functions f, g, u and v, such that

$$E^1 = f\,\mathrm{d}u, \qquad E^2 = g\,\mathrm{d}v. \tag{3.7}$$

We choose (u, v) as coordinates, while the functions f and g remain to be determined by the conditions of the problem. Clearly, with respect to (u, v) we have $\alpha = \alpha(u, v)$, $\beta = \beta(u, v)$ and the eigenvectors E_1, E_2 of $\mathbf{K}$ are given by

$$E_1 = (f)^{-1}\partial_u, \qquad E_2 = (g)^{-1}\partial_v. \tag{3.8}$$

Substituting (3.7) into (3.4), yields

$$\alpha = -(fg)^{-1}\partial_u f, \qquad \beta = (fg)^{-1}\partial_v g. \tag{3.9}$$

Consider again the Hamiltonian function (1.1) in natural (position–momenta, say) coordinates:

$$H = \tfrac{1}{2}g^{ij}p_i p_j + V.$$

In a rigid moving frame in view of the above, we have

$$H = \tfrac{1}{2}g^{ab}p_a p_b + V, \tag{3.10}$$

where $g^{ab} = g^{ij}h^a_i h^b_j$ and $p_a = h_a{}^k p_k$, where $h_a{}^i$ is defined in (1.6) and V is a function of u and v. Next, we apply the vector field $[E_1, E_2]$ to λ_1 and λ_2 to obtain the following integrability conditions:

$$E_1\alpha = -3\alpha\beta, \tag{3.11}$$

$$E_2\beta = 3\alpha\beta. \tag{3.12}$$

Now it is natural to analyze the following three cases defined with respect to α and β.

CI $\alpha = \beta = 0 \Leftrightarrow \lambda_1$ and λ_2 constant,

CII $\alpha = 0, \beta \neq 0 (\alpha \neq 0, \beta = 0) \Leftrightarrow \lambda_1$ constant (λ_2 constant),

CIII $\alpha\beta \neq 0 \Leftrightarrow \lambda_1$ and λ_2 both non-constant.

This classification is intrinsic since the rigid moving frame we are using is defined up to a sign. The general forms of the separable metric

$$\mathrm{d}s^2 = (E^1)^2 + (E^2)^2, \tag{3.13}$$

and the corresponding Killing tensor $\mathbf{K}$ (3.2) will be derived in each case. Having found the Killing tensor, we shall derive the form of the most general separable potential $V(u, v)$

admitted by the original Hamiltonian (1.1). To accomplish this, we take into consideration the condition $\mathrm{d}(\mathbf{B}\,\mathrm{d}V) = 0$ of Theorem 2.1, which may be written in terms of the moving frames as

$$E_1E_2V + 3\beta E_2V - 2\alpha E_1V = 0. \tag{3.14}$$

Once the potential V is found, we derive the second first integral of the Hamiltonian system defined by (1.1) given by $F = K^{ab}p_ap_b + U$ or

$$F(u, v, p_1, p_2) = \lambda_1 p_1^2 + \lambda_2 p_2^2 + U(u, v) \tag{3.15}$$

in the moving frame, by solving the equation $\mathrm{d}U = 2\mathbf{B}\,\mathrm{d}V$. Writing this condition in the moving frame, we immediately obtain the following system

$$E_1U = 2\lambda_1 E_1V, \tag{3.16}$$

$$E_2U = 2\lambda_2 E_2V. \tag{3.17}$$

3.1. *Case I: $\alpha = \beta = 0$*

It follows immediately from (3.9) that $f = f(u)$ and $g = g(v)$. Therefore, $E^1 = f(u)\,\mathrm{d}u$, $E^2 = g(v)\,\mathrm{d}v$, and the metric takes the form

$$\mathrm{d}s^2 = f^2(u)\,\mathrm{d}u^2 + g^2(u)\,\mathrm{d}v^2.$$

We observe that there exist coordinate transformations $(u, v) \to (\tilde{u}, \tilde{v})$, such that

$$E^1 = f(u)\,\mathrm{d}u = \mathrm{d}\tilde{u}, \qquad E^2 = g(u)\,\mathrm{d}u = \mathrm{d}\tilde{u}, \tag{3.18}$$

where

$$\tilde{u} = \int f(u)\,\mathrm{d}u, \qquad \tilde{v} = \int g(v)\,\mathrm{d}v.$$

The remaining coordinate freedom is

$$\tilde{u} = \bar{u} + u_0, \qquad \tilde{v} = \bar{v} + v_0.$$

Thus, for CI we have

$$E^1 = \mathrm{d}u, \qquad E^2 = \mathrm{d}v, \tag{3.19}$$

where the tildes have been dropped. Thus, the metric (3.13) has the form

$$\mathrm{d}s^2 = \mathrm{d}u^2 + \mathrm{d}v^2. \tag{3.20}$$

We conclude that the separable coordinates in this case are *Cartesian*. We also observe, by (3.5), that $R_{1212} = 0$, in CI, which means that the case when both eigenvalues of $\mathbf{K}$ are constant is compatible with only a *flat* two-dimensional Riemannian space. Now taking into account the above facts along with the Killing equation, we easily recover that $\lambda_1 = c_1$ and $\lambda_2 = c_2$, where c_1 and c_2 are constant. Hence,

$$\mathbf{K} = \mathrm{diag}(c_1, c_2), \tag{3.21}$$

and in view of (3.14), we have

$$V(u, v) = V_1(u) + V_2(v). \tag{3.22}$$

Similarly, by making use of (3.16) and (3.17), we find the corresponding U to be

$$U(u, v) = 2kV_1(u) + 2\ell V_2(v). \tag{3.23}$$

We conclude that a second first integral F that is functionally independent of the Hamiltonian H is

$$F(u, v, p_u, p_v) = p_v^2 + 2V_2(v). \tag{3.24}$$

We note that the class of Hamiltonian systems just described has the properties of being bi-Hamiltonian in the separable coordinates (u, v) with respect to the constant Poisson bi-vectors $\mathbf{P}_0$ and $\mathbf{P}_1$:

$$\mathbf{P}_0 = \partial_u \wedge \partial_{p_u} + \partial_v \wedge \partial_{p_v}, \qquad \mathbf{P}_1 = \partial_u \wedge \partial_{p_u} - \partial_v \wedge \partial_{p_v}, \tag{3.25}$$

and having a Lax representation defined by matrices L and M of the form

$$L = \begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix}, \qquad M = \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}, \tag{3.26}$$

where

$$L_i = \begin{pmatrix} \dfrac{1}{\sqrt{2}} p_j & 2w_j \\ \dfrac{f_i(w_j)}{w_j} & -\dfrac{1}{\sqrt{2}} p_j \end{pmatrix}, \qquad M_i = \frac{1}{2w_j} \begin{pmatrix} 0 & 0 \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{p_j}{\sqrt{2}}\right) & -2p_j \end{pmatrix}. \tag{3.27}$$

Here $i, j = 1, 2, i \neq j, w_1 = u, w_2 = v$ and $f_1, f_2 \in C^1(\mathbb{R})$ are arbitrary functions. We note that the separable coordinates (u, v) in this case are simply the *Darboux–Nijenhuis coordinates* [25] defining the *canonical* bi-Hamiltonian structure (3.25). See [26] for more details and illustrative examples.

3.2. *Case II:* $\alpha = 0, \beta \neq 0 (\alpha \neq 0, \beta = 0)$

The condition $\alpha = 0$ in (3.9) immediately yields $f = f(u)$, and, by an appropriate coordinate transformation, we may set $f = 1$. Similarly, we use (3.12) to conclude $\beta = \beta(u)$, which entails in turn after solving (3.12) that $g = C(u)D(v)$, where $C(u)$ and $D(v)$ are arbitrary functions. We may absorb $D(v)$ by a further coordinate transformation to obtain $g = g(u)$. Hence, the metric in this case is given by

$$\mathrm{d}s^2 = \mathrm{d}u^2 + g^2(u)\,\mathrm{d}v^2, \tag{3.28}$$

where $g(u)$ is an arbitrary function. To solve the Killing equation and find the corresponding **K**, we observe that in view of the above $\beta = \partial_u g/g$. Now Eqs. (3.6) transform into the following system of partial differential equations

$$\partial_u\lambda_1 = \partial_v\lambda_1 = \partial_v\lambda_2 = 0, \qquad \partial_u\lambda_2 = \partial_u g g^{-1}(\lambda_2 - \lambda_1). \tag{3.29}$$

Solving for λ_1 and λ_2, we find $\lambda_1 = k$, $\lambda_2 = \ell g^2(u) + k$, where ℓ, k are arbitrary constants. Hence, the Killing tensor in this case takes the form:

$$\mathbf{K} = \mathrm{diag}(k, \ell g^2(u) + k) = k\mathbf{g} + \ell\mathbf{K}_1, \tag{3.30}$$

where $\mathbf{K}_1 = \mathrm{diag}(0, g^2(u))$ and $\mathbf{g}$, $\mathbf{K}_1$ span the two-dimensional Abelian Lie algebra of Killing tensors as in [4]. We note that, since the variable v is ignorable, the Killing tensor $\mathbf{K}_1$ is simply the square of the corresponding Killing vector corresponding to the first integral linear in the momenta.

Remark 3.1. This observation illustrates the fact that Benenti's approach is in fact equivalent to the approach due to Eisenhart [13] and Kalnins and Miller [14]. In the most general case the Killing tensor **K** of Theorem 2.1 is simply a linear combination of the n Killing tensors (including the metric) $\mathbf{g}, \ldots, \mathbf{K_{n-1}}$ in [13,14].

Next, taking into account that $\alpha = 0$ and $f = 1$, we solve Eq. (3.14) for V to obtain

$$V(u, v) = V_1(u) + \frac{V_2(v)}{g^2(u)}, \tag{3.31}$$

where V_1 and V_2 are arbitrary functions. It follows by (3.16) that

$$U(u, v) = 2kV_1(u) + 2\ell V_2(v) + \frac{2kV_2(v)}{g^2(u)}. \tag{3.32}$$

Finally, substituting (3.31) and (3.32) into (3.15) and removing the expression for the Hamiltonian we find a second first integral F for this family of separable Hamiltonian systems just described, namely

$$F(u, v, p_1, p_2) = kg^2(u)p_2^2 + \ell V_1(u) + 2\ell V_2(v) + \frac{kV_2(v)}{g^2(u)}, \tag{3.33}$$

or, in terms of the separable coordinates:

$$F(u, v, p_u, p_v) = c_2 p_v^2 + c_1 V_1(u) + 2c_2 V_2(v) + \frac{c_1 V_2(v)}{g^2(u)}. \tag{3.34}$$

We note that (3.5) in this case becomes

$$R_{1212} = -\partial_u\left(\frac{\partial_u g}{g}\right) - \left(\frac{\partial_u g}{g}\right)^2 = -\frac{g''}{g}, \tag{3.35}$$

or, simply

$$g'' + ag = 0, \tag{3.36}$$

where $a(u) = R_{1212}(u)$. The case $\alpha \neq 0$, $\beta = 0$ corresponds to the metric $\mathrm{d}s^2 = f^2(v)\,\mathrm{d}u^2 + \mathrm{d}v^2$, which can be obtained from (3.28) in an obvious way.

3.3. *Case III: $\alpha\beta \neq 0$*

We begin by proving first that in this case the functions f and g may be assumed equal. Eqs. (3.11) and (3.12) imply that

$$E_1\alpha = -E_2\beta,$$

which, on account of (3.9), may be written as

$$\partial_u\partial_v\left(\ln\left(\frac{f}{g}\right)\right) = 0.$$

It follows $\ln(f/g) = G(u) + H(v)$, where G and H are arbitrary functions, from which we obtain

$$f = g(u, v)C(u)D(v), \tag{3.37}$$

where $C(u) = \mathrm{e}^{G(u)}$ and $D(v) = \mathrm{e}^{H(v)}$. After appropriate coordinate transformations applied to the metric, we get

$$f(u, v) = g(u, v). \tag{3.38}$$

We now proceed to determined the general form of the metric. In view of (3.38), either of (3.11) and (3.12), yields

$$\partial_u\partial_v f^2(u, v) = 0.$$

Therefore,

$$f^2(u, v) = A(u) + B(v), \tag{3.39}$$

where A and B are arbitrary functions. It follows that the metric has the form

$$\mathrm{d}s^2 = (A(u) + B(v))(\mathrm{d}u^2 + \mathrm{d}v^2). \tag{3.40}$$

Remark 3.2. We note immediately that the metric (3.40) is that of the well known *Liouville surface* [27]. Hence, in this case the dynamics of (1.1) can be viewed as the motion of a Liouville surface under the action of a conservative force with potential energy $V(u, v)$.

We proceed to find the corresponding Killing tensor $\mathbf{K}$. Substituting (3.9) along with (3.39) into (3.6) leads to the following system of partial differential equations with respect to λ_1 and λ_2:

$$\partial_u\lambda_1(u, v) = \partial_v\lambda_2(u, v) = 0, \qquad \partial_v\lambda_1(u, v) = \frac{B'(v)}{A(u) + B(v)}(\lambda_1(u, v) - \lambda_2(u, v)),$$
$$\partial_u\lambda_2(u, v) = \frac{A'(u)}{A(u) + B(v)}(\lambda_2(u, v) - \lambda_1(u, v)). \tag{3.41}$$

Solving (3.41), we obtain $\lambda_1 = kB(v)+\ell$ and $\lambda_2 = -kA(u)+\ell$, where k and ℓ are arbitrary constants. Thus

$$\mathbf{K} = \mathrm{diag}(kB(v) + \ell, -kA(u) + \ell) = \ell\mathbf{g} + k\mathbf{K}_1, \tag{3.42}$$

where $\mathbf{K}_1 = \mathrm{diag}(B(v), -A(u))$ (see Remark 3.1). Eq. (3.14) for $V(u, v)$ may be written as

$$\partial_u\partial_v[(A(u) + B(v))V(u, v)] = 0,$$

which has the solution

$$V(u, v) = \frac{V_1(u) + V_2(v)}{A(u) + B(v)}, \tag{3.43}$$

where V_1 and V_2 are arbitrary functions. It follows that (3.16) and (3.17) may be solved to obtain

$$U(u, v) = 2\ell V(u, v) + 2k\frac{B(v)V_1(u) - A(u)V_2(v)}{A(u) + B(v)}. \tag{3.44}$$

We conclude that the second first integral independent of H has the form

$$F(u, v, p_1, p_2) = B(v)p_1^2 - A(u)p_2^2 + 2\left(\frac{B(v)V_1(u) - A(u)V_2(v)}{A(u) + B(v)}\right). \tag{3.45}$$

Noting that ${h_1}^1 = f^{-1}$, ${h_2}^2 = f^{-1}$, ${h_1}^2 = {h_2}^1 = 0$, we may rewrite (3.45) in terms of the coordinates as

$$F(u, v, p_u, p_v) = \frac{B(v)(p_u^2 + 2V_1(u)) - A(u)(p_v^2 + 2V_2(v))}{A(u) + B(v)}. \tag{3.46}$$

We note that the form of the Hamiltonian H (1.1) in the coordinates (u, v) becomes

$$H(u, v, p_u, p_v) = \frac{p_u^2 + p_v^2}{2(A(u) + B(v))} + \frac{V_1(u) + V_2(v)}{A(u) + B(v)}. \tag{3.47}$$

The forms (3.46) and (3.47) demonstrate that the Hamiltonian system under consideration is a Liouville system [28] in the separable coordinates (u, v). Conversely, it is easy to see that the Hamilton–Jacobi equation corresponding to (3.47) separates in the coordinates (u, v). Indeed, in this case (1.3) takes the following form:

$$\frac{1}{2(A(u) + B(v))}((\partial_u W)^2 + (\partial_v W)^2 + 2(V_1(u) + V_2(v))) = E.$$

Now, putting $W(u, v) = W_1(u) + W(v)$, we find the complete integral W to be

$$W(u, v) = \int \sqrt{\beta - 2V_1(u) + EA(u)}\,\mathrm{d}u + \int \sqrt{-\beta - 2V_2(v) + EB(v)}\,\mathrm{d}v.$$

Differentiating W with respect to β and E, we can find the solutions for specific choices of $A(u)$, $B(v)$, $V_1(u)$ and $V_2(v)$. Hence, without imposing any restriction on the curvature of the corresponding pseudo-Riemannian manifold we have proven the following criterion of separability.

Theorem 3.3. *The following conditions are equivalent.*

1. *The Riemannian manifold* $(\tilde{M}, \mathbf{g})$ *defined by* (3.48) *admits a valence two Killing tensor* $\mathbf{K}$ *with distinct eigenvalues;*
2. *There exist coordinates* (u, v) *with respect to which the metric takes the form* (3.40) ;
3. *The Hamiltonian system defined by the Hamiltonian*

$$H = \tfrac{1}{2} g^{ij}(\mathbf{q}) p_i p_j + V(\mathbf{q}), \quad i, j = 1, 2 \tag{3.48}$$

in the Riemannian manifold $(\tilde{M}, \mathbf{g})$ *of an arbitrary curvature can be integrated by separation of variables.*

Remark 3.4. The (2) ⇔ (3) part of Theorem 3.3 was proven first in 1881 using local coordinates by Morera [29], who also extracted the four separable systems of coordinates in the Euclidean flat space (see below). The (1) ⇔ (3) part is simply a restatement of Theorem 2.1 and (1) ⇔ (2) follows from the above considerations.

Having derived the explicit formula (3.46) for a second first integral F, we can now investigate whether or not the Liouville system (3.47) admits a bi-Hamiltonian representation with respect to the coordinates (u, v). Recall that the bi-Hamiltonian property is a combination of algebraic and differential conditions, which can be quite restrictive for low-dimensional Hamiltonian systems. Indeed, it is easy to see that the symplectic form $\boldsymbol{\omega}_1$ corresponding to F: $i_{X_H}\boldsymbol{\omega}_1 = -\mathrm{d}F$ is given by

$$\boldsymbol{\omega}_1 = 2B(v)\,\mathrm{d}u \wedge \mathrm{d}p_u - 2A(u)\,\mathrm{d}v \wedge \mathrm{d}p_v. \tag{3.49}$$

Clearly, (3.49) satisfies the differential conditions $\mathrm{d}\boldsymbol{\omega}_1 = 0$ and $L_{X_H}(\boldsymbol{\omega}_1) = 0$ iff $A(u) = B(v) = \text{const}$. In this case $\boldsymbol{\omega}_1$ is equivalent to $\mathbf{P}_1$ in (3.25). Therefore, taking into account the result of Theorem 1 in [26], we answer the question of whether there exists a second Hamiltonian representation with respect to F by the following result.

Proposition 3.5. *The Liouville system defined by* (3.47) *admits a bi-Hamiltonian representation in the separable coordinates* (u, v) *iff the coordinates are Cartesian.*

Finally, we note that the formula (3.5) assumes in this case the following form.

$$R_{1212} = -\frac{1}{f}\left[\partial_u\left(\frac{\partial_u f}{f^2}\right) + \partial_v\left(\frac{\partial_v f}{f^2}\right)\right] - \left(\frac{\partial_u f}{f^2}\right)^2 - \left(\frac{\partial_v f}{f^2}\right)^2, \tag{3.50}$$

where $f^2(u, v) = A(u) + B(v)$.

4. Applications

We now apply the classifications of the previous section to Hamiltonian systems defined in particular Riemannian spaces.

4.1. Two-dimensional Euclidean space E^2

In this case $R_{1212} = 0$, which entails

$$E_2\alpha - E_1\beta = \alpha^2 + \beta^2.$$

Consider now the following three separable cases (SC), defined with respect to the functions α and β.

SCI: $\alpha = \beta = 0$.

In this case the separable coordinates are obviously Cartesian and $R_{1212} = 0$, is automatically satisfied.

SCII: $\alpha = 0$, $\beta \neq 0$.

Solving Eq. (3.36) we obtain that the metric can be written as follows:

$$\mathrm{d}s^2 = \mathrm{d}u^2 + u^2\,\mathrm{d}v^2, \tag{4.1}$$

which we immediately recognize as the Euclidean metric in polar coordinates.

SCIII: $\alpha\beta \neq 0$.

Employing (3.5) ($\alpha\beta \neq 0$, $R_{1212} = 0$) to find the functions $A(u)$ and $B(v)$ defining the formula for the metric of a Liouville surface, we arrive at the following equation.

$$(A(u) + B(v))(A''(u) + B''(v) = (A'(u))^2 + (B'(v))^2,$$

which after taking partial derivatives reduces to

$$\frac{A'''(u)}{A'(u)} + \frac{B'''(v)}{B'(v)} = k^2 \tag{4.2}$$

for some constant $k \geq 0$. Solving (4.2) separately for $k = 0$ and $k \neq 0$ yields the metrics

$$\mathrm{d}s^2 = (u^2 + v^2)(\mathrm{d}u^2 + \mathrm{d}v^2), \tag{4.3}$$

and

$$\mathrm{d}s^2 = a^2(\cosh^2(u) - \cos^2(v))(\mathrm{d}u^2 + \mathrm{d}v^2), \tag{4.4}$$

respectively, where a is a scaling parameter. We note that the expressions (4.3) and (4.4) represent the Euclidean metric in parabolic and elliptic–hyperbolic coordinates, where a represents half the distance between the focii. Hence, we have extracted the four separable systems of coordinates in the Euclidean space by employing the method of moving frames. The corresponding Killing tensors, second first integrals and potential functions can be recovered by making use of the formulas derived in Section 3.

4.2. Surfaces of rotation

A surface of rotation is the surface generated by the rotation of a plane curve C around an axis in its plane. If C is parametrized by the equations $\rho = \rho(u)$ and $z = z(u)$, the position vector of the surface of rotation is $\mathbf{r} = \{\rho(u)\cos v, \rho(u)\sin v, z(u)\}$, where u is the parameter of the curve C, ρ is the distance between a point on the surface and the axis z of rotation and v is the angle of rotation, which is the ignorable (cyclic) coordinate. The metric of the surface of rotation is

$$\mathrm{d}s^2 = ((\rho')^2 + (z')^2)\,\mathrm{d}u^2 + \rho^2\,\mathrm{d}v^2. \tag{4.5}$$

Clearly, the metric (4.5) can be reduced to the form (3.28) by an appropriate coordinate transformation. Once the curvature $R_{1212}(u)$ is known, the function(s) $g(u)$ and the corresponding metric(s) may be recovered from (3.36) and vice versa. Consider an example. The metric

$$\mathrm{d}s^2 = a^2\,\mathrm{d}u^2 + \ell^2\left(1 + \frac{a}{\ell}\cos u\right)^2\mathrm{d}v^2 \tag{4.6}$$

defines the surface of a two-dimensional torus T^2, where a and ℓ are the radii of the rotating and axial circles, respectively. We note that in this paper we do not consider global properties of two-dimensional pseudo-Riemannian manifolds, hence here T^2 is not a topological torus. Locally, the metric (4.6) yields one system of separable coordinates with $g(u) = \ell(1 + (a/\ell)\cos(u/a)$, $R_{1212} = \cos(u/a)/(a\ell + a\cos(u/a))$ and the other quantities as in Case II of Section 3 corresponding to the given $g(u)$.

4.3. Surfaces of constant curvature

In this section, we assume the curvature $R_{1212} = \epsilon a^2$, where $\epsilon = \pm 1$ and $a > 0$ is constant. Let us consider again the two cases: $\alpha = 0$, $\beta \neq 0$ and $\alpha\beta \neq 0$.

Case I: $\alpha = 0$, $\beta \neq 0$. In this case the coordinate v is ignorable (cyclic). Solving (3.36) for $a(v) = \text{const}$, yields:

$$g(u) = c_1\cos au + c_2\sin au = \tilde{c}_1\cos au + \tilde{c}_2\frac{1}{a}\sin u, \quad \epsilon = 1,$$

$$g(u) = c_3\,\mathrm{e}^{au} + c_4\,\mathrm{e}^{au} = \tilde{c}_3\cosh au + \tilde{c}_4\frac{1}{a}\sinh au, \quad \epsilon = -1.$$

Now varying the constants of integration we recover four distinct solutions for $g(u)$ corresponding to the following metrics.

$$\mathrm{d}s^2 = \frac{1}{a}(\mathrm{d}u^2 + \sin au\,\mathrm{d}v^2), \quad \epsilon = 1, \tag{4.7}$$

$$\mathrm{d}s^2 = \mathrm{d}u^2 + \cosh^2 au\,\mathrm{d}v^2, \quad \epsilon = -1, \tag{4.8}$$

$$\mathrm{d}s^2 = \mathrm{d}u^2 + \left(\frac{\sinh au}{a}\right)^2\mathrm{d}v^2, \tag{4.9}$$

$$\mathrm{d}s^2 = \mathrm{d}u^2 + \mathrm{e}^{-2au}\,\mathrm{d}v^2. \tag{4.10}$$

Using the explicit expression for the function $g(u)$ above and the formulas (3.30), (3.31) and (3.33) we can write down in each case the corresponding potentials, Killing tensors and second first integrals.

Case II: $\alpha\beta \neq 0$. Again, assume $R_{1212} = \epsilon a^2$. Then (3.50) reads

$$(A + B)(A'' + B'') - (A')^2 - (B')^2 = -2\epsilon a^2 (A + B)^3, \tag{4.11}$$

where $A = A(u)$ and $B = B(v)$. Eq. (4.11) can be separated as follows:

$$\frac{A'''}{A'} + 12\epsilon a^2 A = -\frac{B'''}{B'} - 12\epsilon a^2 B = \lambda.$$

Hence, we arrive at the following two equations for A and B, respectively,

$$A''' + 12\epsilon a^2 AA' = \lambda A', \qquad B''' + 12\epsilon a^2 BB' = -\lambda B'. \tag{4.12}$$

Assuming $\lambda \neq 0$ and solving (4.12) with respect to u and v, we get

$$\pm\mathrm{d}u = \frac{\mathrm{d}A}{(-4\epsilon a^2 A^3 + \lambda A^2 + 2\ell A + 2m)^{1/2}}, \tag{4.13}$$

$$\pm\mathrm{d}v = \frac{\mathrm{d}B}{(-4\epsilon a^2 B^3 - \lambda B^2 + 2\tilde{\ell} B + 2\tilde{m})^{1/2}}, \tag{4.14}$$

where ℓ, $\tilde{\ell}$, m and $\tilde{m}$ are the constants of integration. Substituting the corresponding expressions for $(A')^2$ and $(B')^2$ into (4.11), we easily find $\ell = \tilde{\ell}$ and $m = -\tilde{m}$. Next, substituting (4.13) and (4.14) into (3.40), then factoring out $-1/(4\epsilon a^2)$ and changing the variables: $A \to \tilde{A} + \lambda/12\epsilon a^2$, $B \to -\tilde{B} - \lambda/12\epsilon a^2$, we arrive after dropping tildes at the following metric:

$$\mathrm{d}s^2 = -\frac{1}{4\epsilon a^2}(A - B)\left(\frac{\mathrm{d}A}{p_3(A)} - \frac{\mathrm{d}B}{p_3(B)}\right), \tag{4.15}$$

where $p_3(x) = x^3 + px + q$ with arbitrary coefficients p and q. Note that we have derived the metric (4.15) without solving (4.13) and (4.14) for A and B, respectively. Comparing the metrics (3.40) and (4.15) we see that the latter metric is not in the Liouville form and so we cannot complete the analysis by deriving the corresponding first integrals, potentials and Killing tensors. However, since the functions A and B and their derivatives in (4.13) and (4.14) essentially parametrize appropriate elliptic curves, clearly it can be done by expressing A and B in terms of the Weierstrass function $\wp$. Indeed, by appropriate linear transformations Eqs. (4.13) and (4.14) can be transformed into the corresponding form of the Weierstrass differential equation

$$\left(\frac{\mathrm{d}\wp}{\mathrm{d}z}\right)^2 = 4\wp^3 - g_2\wp - g_3,$$

thus leading to the following solutions for the functions $A(u)$ and $B(v)$, respectively:

$$A(u) = \wp(a\sqrt{-\epsilon}u + c_1; \omega_1, \omega_2) - \lambda, \tag{4.16}$$

$$B(v) = \wp(a\sqrt{-\epsilon}v + c_2; \omega_1, -\omega_2) + \lambda, \tag{4.17}$$

where c_1, c_2, λ are arbitrary functions and ω_1, ω_2 define the periods of the meromorphic, doubly periodic function $\wp$. Now we can use the expressions (4.16) and (4.17) and the analysis of Section 3 to derive in each case the corresponding separable potential (formula (3.43)), Killing tensor (formula (3.42)), as well as the second first integral (formula (3.45)).

Let x_1, x_2 and x_3 be the roots of p_3: $p_3(x) = (x - x_1)(x - x_2)(x - x_3)$. Without loss of generality we impose the condition $A > B$. To extract all the metrics depending on different choices of x_1, x_2 and x_3, we impose the condition that the right-hand side of (4.15) must be positive definite. When $\epsilon = 1$ there is only one possibility for A and B for which (4.15) is positive definite, while $\epsilon = -1$ leads to six different possibilities:

$$x_1 < B < x_2 < A < x_3, \quad \epsilon = 1, \tag{4.18}$$

$$x_1 < x_2 < B < x_3 < A, \quad \epsilon = -1, \tag{4.19}$$

$$B < x_1 < x_2 < x_3 < A, \tag{4.20}$$

$$B < x_3 < A, \quad x_1 = \bar{x}_2, \tag{4.21}$$

$$x_1 = x_2 < B < x_3 < A, \tag{4.22}$$

$$B < x_1 = x_2 < x_3 < A, \tag{4.23}$$

$$B < x_1 = x_2 = x_3 < A. \tag{4.24}$$

We observe that these separable cases were first derived by Olevsky [30], while studying separability of Laplace–Beltrami's operator in the spaces of constant curvature. He used Eisenhart's (coordinate) approach to the problem. The moving frame method applied to two-dimensional separable Hamiltonian systems yields the same results without considering initially a particular system of coordinates. We note that the separable coordinates (A, B) are essentially the eigenvalues of the Killing tensor $\mathbf{K}_1$ in (3.42).

5. Concluding remarks

The method of moving frames employed in this paper is apparently the first application of the method to finite-dimensional Hamiltonian systems with a potential. We note also that the conclusion about separability of the corresponding Hamilton–Jacobi equation here follows directly from the integrability conditions for the Killing tensor equations. In contrast, the earlier results due to Eisenhart [9] (where a version of this method adapted to local coordinate was used to separate geodesic equation), rested on the Stäckel criterion of separability. When combined with Benenti's criterion of orthogonal separability of the Hamiltonian system defined by (1.1) the method provides a powerful tool to classify (locally) separable cases for Hamiltonian systems defined by (1.1) in pseudo-Riemannian manifolds of arbitrary curvature. Admittedly, it can be used with the most benefit in the spaces of low dimensions. As an illustration we have presented a comprehensive study of separability in two-dimensional Riemannian manifolds. Although the approach in this case allows us mainly to recover known results, this work commences a program of studying orthogonal separability in three- and four-dimensional pseudo-Riemannian manifolds. The latter case is very important for investigating Hamiltonian systems of General Relativity.

The method can also be used to study *super-separability* [18] of (1.1), which is the property of the potential V being compatible (i.e., satisfy $\mathrm{d}(\mathbf{B}\,\mathrm{d}V) = 0$) in more than one separable system. Thus, we can determine the form of the super-separable potentials by considering pairs of separable coordinate systems. Specifically, we impose the compatibility condition of both coordinate systems using the coordinate transformation between them. For example, the potentials separable in Cartesian coordinates have the form (3.22). Transforming (3.22) to polar coordinates, we get $\tilde{V}(u, v) = V_1(u \cos v) + V_2(u \sin v)$. Now impose the condition $\mathrm{d}(\mathbf{B}\,\mathrm{d}\tilde{V}) = 0$ in polar coordinates:

$$\partial_u(u^2 \partial_v \tilde{V}) = 0. \tag{5.1}$$

Solving (5.1) and going back to Cartesian coordinates, we easily obtain

$$V(x, y) = \ell(x^2 + y^2) + \frac{m}{x^2} + \frac{n}{y^2}, \tag{5.2}$$

where ℓ, m and n are constants, which is the general potential separable in both Cartesian and polar coordinates, first found by Friš et al. in [31], while studying (product) separability of field equations. This procedure can be used in pseudo-Riemannian manifolds of arbitrary curvature, once all separable coordinate systems of coordinates have been determined.

Finally, employing the method of moving frames has also allowed us to demonstrate the equivalence between the Kalnins and Miller theory of orthogonal separability [14] and the theory of Benenti [6]. As we have seen, the Killing tensor $\mathbf{K}$ defining the linear operator $\mathbf{B}$ in Theorem 2.1 is in fact a linear combination of the n basic Killing tensors (including the metric $\mathbf{g}$) of Theorem 6 in [14]. (See Remark 3.1 and the formulas (3.21), (3.30) and (3.42).)

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